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The stability boundaries of the steady motion of a satellite with a gyroscope☆

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ABSTRACT

Parts of the asymptotic stability boundaries of the uniform motion of the centre of mass of a system of bodies consisting of an asymmetrical satellite with a three-axis gyroscope in a circular orbit are investigated by the second Lyapunov method. Terms of the Lyapunov function that are higher than the second order are enlisted for the investigation. The sign-definiteness criterion of inhomogeneous forms is employed for the corresponding function. Parts of the stability boundaries in which the steady motion investigated is asymptotically stable are established using the Lyapunov asymptotic stability theorem. Application of the Barbashin and Krasovskii theorems reveals parts of the stability boundaries in which the steady motion is unstable. It is established that the asymptotic stability of the steady motion investigated is solved by expanding the Lyapunov function to sixth-order terms.

1. Introduction

The necessary conditions for the asymptotic stability of the uniform motion of the centre of mass of a system of bodies consisting of an asymmetrical satellite and a three-axis gyroscope in a circular orbit with a constant angular velocity ω_0 have been obtained.[1] The orbital system of coordinates $OX_1X_2X_3$, in which the OX_1 axis is directed along a tangent to the orbit, the OX_3 axis is directed along a radius vector from the centre of the Earth, and the OX_2 axis completes a right system of coordinates, is introduced to describe the satellite motion. The system of coordinates attached to the satellite $Ox_1x_2x_3$ is directed along the principal axes of inertia of the satellite with corresponding moments of inertia A, B and C. We will use ω_i (i = 1, 2, 3) to denote the projections of the absolute angular velocity of the system of bodies onto the Ox_i axes, and we will use H_i (i = 1, 2, 3) to denote the projections of the angular momentum onto the Ox_i axes. The orientation of the $Ox_1x_2x_3$ system of coordinates relative to $OX_1X_2X_3$ is specified by the angles γ, δ and β,[1] so that the projections of the unit vectors OX_2 and OX_3 onto the Ox_1, Ox_2 and Ox_3 axes are as follows:

$$a_{21} = \sin\beta,\quad a_{22} = \cos\beta\cos\gamma,\quad a_{23} = -\cos\beta\sin\gamma,\quad a_{31} = -\cos\delta\cos\beta$$
$$a_{32} = -\sin\delta\sin\gamma + \cos\delta\sin\beta\cos\gamma,\quad a_{33} = -\sin\delta\cos\gamma - \cos\delta\sin\beta\sin\gamma$$

The equations of motion of the satellite system of bodies have the form[1]

$$A\dot{\omega}_1 + (C-B)\omega_2\omega_3 + \dot{H}_1 + \omega_2H_3 - \omega_3H_2 = 3\omega_0^2(C-B)a_{32}a_{33}$$
$$(ABC,\quad H_1H_2H_3,\quad \omega_1\omega_2\omega_3,\quad a_{31}a_{32}a_{33})$$
$$\dot{\gamma} = \omega_1 - (\omega_2\cos\gamma - \omega_3\sin\gamma)\mathrm{tg}\beta,\quad \dot{\beta} = \omega_2\sin\gamma + \omega_3\cos\gamma$$
$$\dot{\delta} = -\omega_0 + \frac{(\omega_2\cos\gamma - \omega_3\sin\gamma)}{\cos\beta}$$
$$\tau_1\dot{H}_1 + H_1 = J_1\omega_1,\quad \tau_2\dot{H}_2 + H_2 - H = J_2(\omega_2 - \omega_0),\quad \tau_3\dot{H}_3 + H_3 = J_3\omega_3 \tag{1.1}$$

where τ_i (i = 1, 2, 3) denotes certain positive constants. The moments of inertia of the gyroscope in the $Ox_1x_2x_3$ system have the values J_i (i = 1, 2, 3). The angular momenta of the gyroscope are treated as controls. The system of equations (1.1) admits of steady motion of the

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form[1]

$$\omega_1 = \omega_3 = \gamma = \delta = \beta = H_1 = H_3 = 0, \quad \omega_2 = \omega_0 = \text{const}, \quad H_2 = H = \text{const} \tag{1.2}$$

The stability of motion (1.2) was investigated using the function[1]

$$V(\omega_1,\omega_2,\omega_3,\gamma,\delta,\beta,H_1,H_2,H_3) = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) + \frac{1}{2}\omega_0^2(B-3A)$$

$$+\omega_0(H_2 - H_1a_{21} - H_2a_{22} - H_3a_{23} - A\omega_1a_{21} - B\omega_2a_{22} - C\omega_3a_{23})$$

$$+\frac{3}{2}\omega_0^2\left\langle Aa_{31}^2 + Ba_{32}^2 + Ca_{33}^2\right\rangle + \frac{1}{2}\left\langle \frac{H_1^2}{J_1} + \frac{(H_2-H)^2}{J_2} + \frac{H_3^2}{J_3}\right\rangle$$

By virtue of equations of motion (1.1), it has the derivative of constant sign

$$\dot{V} = -\left\langle \frac{\tau_1}{J_1}\dot{H}_1^2 + \frac{\tau_2}{J_2}\dot{H}_2^2 + \frac{\tau_3}{J_3}\dot{H}_3^2\right\rangle$$

For the equations of perturbed motion we introduce the deviations

$$x_1 = \omega_1, \quad x_2 = \omega_2 - \omega_0, \quad x_3 = \omega_3, x_4 = \gamma, \quad x_5 = \delta, \quad x_6 = \beta, x_7 = H_1,$$
$$x_8 = H_2 - H, \quad x_9 = H_3$$

Then, the following function can serve as the Lyapunov function for investigating the stability of the zero solution of the equations of perturbed motion

$$\bar{V}(x) = V(x_1, \omega_0 + x_2, x_3, x_4, x_5, x_6, x_7, H + x_8, x_9)$$

An investigation of the stability of motion (1.2) based on conditions for the sign-definiteness of the quadratic part of the function $\bar{V}(x)$ has been performed.[1] The asymptotic stability region is specified by the inequalities

$$\omega_0[4(B-A) - J_1] + H > 0, \quad \omega_0(B - C - J_3) + H > 0, \quad C > A \tag{1.3}$$

It was established[1] that there are no complete trajectories of the set $\dot{V} = 0$ for equations of motion (1.1) when $B \neq A$. In such a case, according to the Barbashin–Krasovskii theorem,[2,3] motion (1.2) is asymptotically stable. Instability of motion (1.2) was demonstrated in Ref. 1 using the Barbashin and Krasovskii theorems when the quadratic part of $\bar{V}(x)$ is sign-variable.

In order to investigate the boundaries of the necessary conditions for asymptotic stability, each of the parts of stability boundaries (1.3) will be studied below. The direct Lyapunov method with the Lyapunov function $\bar{V}(x)$ will be used.[3,4]

2. Investigation of the stability boundaries

In the general case, on the stability boundaries, it is necessary to analyse the sign-definiteness of an inhomogeneous function of the form

$$W(x) = W_{2m}(x_1,\ldots,x_n) + W_*(x_1,\ldots,x_{n+l}), \quad x \in R^{n+l}, \quad n,l,m \geq 1 \tag{2.1}$$

Here $W_{2m}(x_1, \ldots, x_n)$ is a form of the lowest order $2m$ that is positive-definite for its variables, and $W_*(x)$ is a polynomial consisting of terms of degrees higher than $2m$. The real solutions $W(x) = 0$ in the vicinity of the origin of coordinates can be sought in the form of the parametric branches[5,6]

$$x_i = \sum_{|p|=L}^{\infty} b_{i(p)}t^p, i = 1,\ldots,n; \quad b_{i(p)} \in R, \quad L > M, \quad x_{n+1} = \delta_1 t_1^M, \ldots, \quad x_{n+l} = \delta_l t_l^M$$
$$t^p = t_1^{p_1} \times t_2^{p_2} \times \ldots \times t_l^{p_l}; \quad |p| = p_1 + p_2 + \ldots + p_l, \quad p_j \geq 0, \quad j = 1,\ldots,l \tag{2.2}$$

where p_j denotes non-negative integer exponents, the value $\delta_j = -1$ is taken only for even M when $x_{n+j} < 0$, while $\delta_j = +1$ is taken in the remaining cases, and positive integer values of L and M are selected when constructing the solutions $W(x) = 0$. Substituting expressions (2.2) into function (2.1) we obtain the series

$$W(x(t)) = W_1(t) = A_Q(b_{i(p)}; M; L; t) + \ldots$$

where $A_Q(b_{i(p)}; M; L; t)$ is a form of the lowest order Q in the parameter t. For a certain L, suppose the fraction Q/M is reduced to the irreducible fraction q/m. Then the sign-definiteness of the function $W(x)$ is resolved by the following theorem.

Theorem 1. *If a) $q = 2\alpha + 1$ (α is an integer) or b) $q = 2\alpha$ and $A_Q(b_{i(p)}; M; L; t)$ is a sign-definite form for certain real $b_{i(p)}$, the function $W(x)$ is sign-definite.*

If $q = 2\alpha$ and $A_Q(b_{i(p)}; M; L; t)$ is a positive-definite form for all $b_{i(p)} \in R$, the function $W(x)$ is positive-definite.

If $q = 2\alpha$ and $A_Q(b_{i(p)}; M; L; t)$ is a form of constant sign for all $b_{i(p)} \in R$, the function $W(x)$ can be sign-definite or sign-variable for terms of order higher than Q.

Theorem 2. *When the sign-definiteness of form (2.1) with m = 1 is analysed, in expansion (2.2) we can assume at once that*

$$M = 1, \delta_j = 1, j = 1, \ldots, l$$

The investigation of the sign-definiteness of such forms has previously been described in greater detail.[7,8]

As the Lyapunov function for the equations of perturbed motion we take the function $\bar{V}(x)$, whose sign-definiteness should be determined by terms up to a finite order.[8] Therefore, we expand the trigonometric functions in the expressions for a_{ij} ($i = 2, 3$; $j = 1, 2, 3$) into Maclaurin series and take into account terms up to the sixth order. To simplify the analysis of the polynomials, we first disregard the variables

$$y_9 = x_2 + \omega_0(1 - a_{22}); \quad y_8 = x_8 + J_2\omega_0(1 - a_{22})$$

Then the variables y_8 and y_9 appear in the expression for $\bar{V}(x(y))$ only in the form of the second-degree terms $(By_9^2 + y_8^2/J_2)$, and they can be excluded from the investigation of the sign-definiteness of $\bar{V}(x)$. After subsequently redefining $y_j = x_j$ ($j = 1, 3, 4, \ldots, 7$), we expand the function $\bar{V}(x)$ to sixth-order terms:

$$V_1(y) = \bar{V}(x(y)) = \frac{1}{2}[Ay_1^2 + y_2^2/J_3 + Cy_3^2 + \omega_0(H + \omega_0(B + J_2))y_4^2 + 3\omega_0^2(C - A)y_5^2$$

$$+ \omega_0(H + \omega_0(2B - 3A + J_2))y_6^2 + y_7^2/J_1] + \omega_0(y_2y_4 - Ay_1y_6 - y_6y_7)$$

$$+ Cy_3y_4 + 3\omega_0^2(C - B)y_4y_5y_6 + \frac{1}{6}\omega_0y_6^2(Ay_1y_6 - 3y_2y_4 - 3Cy_3y_4)$$

$$-\frac{1}{6}\omega_0y_4^3(y_2 + Cy_3) - \frac{1}{6}\omega_0[H + \omega_0(B + J_2)]y_4^4 + \frac{3}{2}\omega_0^2(B - C)y_4^2y_5^2$$

$$+\frac{1}{4}\omega_0[\omega_0(6C - 8B - 2J_2) - H]y_4^2y_6^2 + \frac{1}{2}(A - C)\omega_0^2y_5^4 + \frac{3}{2}\omega_0^2(A - B)y_5^2y_6^2$$

$$+\frac{1}{24}\omega_0[\omega_0(12A - 16B - 4J_2) - H]y_6^4 + \frac{1}{6}\omega_0y_6^3y_7 + \frac{1}{2}(B - C)y_4y_5y_6(4y_4^2 + 4y_5^2 - y_6^2)$$

$$-\frac{1}{120}\omega_0y_6^5(Ay_1 + y_7) + \frac{1}{120}\omega_0y_4(y_2 + Cy_3)(y_4^4 + 10y_4^2y_6^2 + 5y_6^4)$$

$$+\frac{1}{720}\omega_0[H + 16\omega_0(B + J_2)]y_4^6 + \frac{1}{48}\omega_0[H + 8\omega_0(4B - 3C + J_2)]y_4^2y_6^2(y_4^2 + y_6^2)$$

$$+\frac{1}{2}\omega_0^2(C - B)y_4^2y_5^2(y_5^2 + 3y_6^2) + \frac{1}{15}\omega_0^2(C - A)y_5^6 + \frac{1}{2}\omega_0^2(B - A)y_5^2y_6^2(y_5^2 + y_6^2)$$

$$+\frac{1}{720}\omega_0[H + 16\omega_0(4B - 3A + J_2)]y_6^6$$

The quadratic part of the function $V_1(y)$ can be represented in the form of the sum

$$\frac{1}{2}(y_1, y_6, y_7)S_1(y_1, y_6, y_7)' + \frac{1}{2}(y_3, y_4, y_2)S_2(y_3, y_4, y_2)' + \frac{3}{2}\omega_0^2(C - A)y_5^2 + \frac{1}{2}(By_9^2 + y_8^2/J_2)$$

where

$$S_1 = \begin{Vmatrix} A & -A\omega_0 & 0 \\ -A\omega_0 & (4B - 3A)\omega_0^2 + H\omega_0 & -\omega_0 \\ 0 & -\omega_0 & 1/J_1 \end{Vmatrix}, \quad S_2 = \begin{Vmatrix} C & C\omega_0 & 0 \\ C\omega_0 & B\omega_0^2 + H\omega_0 & \omega_0 \\ 0 & \omega_0 & 1/J_3 \end{Vmatrix}$$

The analysis of the equation

$$A_Q(b_{i(p)}; M; L; t) = 0 \tag{2.3}$$

will be considerably simpler after converting the matrices S_1 and S_2 into diagonal matrices. For this purpose, we write the characteristic equations

$$f_1(m) = \det(S_1 - mE) = \frac{1}{J_1}\{J_1 m^3 - [J_1(A + H\omega_0 + (4B - 3A)\omega_0^2) + 1]m^2$$

$$+ [(4ABJ_1 - 2A^2J_1 + 4B - 3A - J_1)\omega_0^2 + H(AJ_1 + 1)\omega_0 + A]m$$

$$- A\omega_0[(4B - 4A - J_1)\omega_0 + H]\} = 0,$$

$$f_2(k) = \det(S_2 - kE) = \frac{1}{J_3}\{J_3 k^3 - [J_3\omega_0(B\omega_0 + H) + CJ_3 + 1]k^2$$

$$+ [(B - J_3)(CJ_3 + 1)\omega_0^2 + H\omega_0(CJ_3 + 1) + C]k - C\omega_0[(B - C - J_3)\omega_0 + H]\} = 0$$

Here and in what follows m_i and k_i (i = 1, 2, 3) denote the eigenvalues of the matrices S_1 and S_2.

In the region specified by the first inequality in (1.3), all the eigenvalues m_i (i = 1, 2, 3) of the matrix S_1 are positive by virtue of the positive-definiteness of the quadratic form $(y_1, y_6, y_7)S_1(y_1, y_6, y_7)'$. On the boundary of this region, at least one of the m_i vanishes. In the region specified by the second inequality in (1.3), all the eigenvalues of the equation $f_2(k) = 0$ are also positive, and on the boundary of this region, one of the k_i vanishes.

The matrices of the corresponding linear transformations

$$T_1 = \begin{Vmatrix} \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 \\ 1 & 1 & 1 \\ \tilde{J}_{11} & \tilde{J}_{12} & \tilde{J}_{13} \end{Vmatrix}, \quad T_2 = \begin{Vmatrix} \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 \\ -1 & -1 & -1 \\ \tilde{J}_{31} & \tilde{J}_{32} & \tilde{J}_{33} \end{Vmatrix}$$

perform the required transformation of the variables singled out

$$(y_1, y_6, y_7)' = T_1(z_1, z_6, z_7)', \quad (y_3, y_4, y_2)' = T_2(z_3, z_4, z_2)'$$

when the notation introduced is defined as follows:

$$\tilde{A}_i = \frac{A\omega_0}{A - m_i}, \quad \tilde{C}_i = \frac{C\omega_0}{A - k_i}, \quad \tilde{J}_{1i} = \frac{J_1\omega_0}{1 - J_1 m_i}, \quad \tilde{J}_{3i} = \frac{J_3\omega_0}{1 - J_3 k_i}$$

Then the quadratic form

$$(y_1, y_6, y_7)S_1(y_1, y_6, y_7)' + (y_3, y_4, y_2)S_2(y_3, y_4, y_2)'$$

is reduced to the sum of squares

$$M_1 z_1^2 + K_1 z_3^2 + K_2 z_4^2 + M_2 z_6^2 + M_3 z_7^2 + K_3 z_2^2$$

where

$$M_i = m_i^2\omega_0^2\left[\frac{A}{(A - m_i)^2} + \frac{J_1^3}{(1 - J_1 m_i)^2}\right], \quad K_i = k_i^2\omega_0^2\left[\frac{C}{(C - k_i)^2} + \frac{J_1^3}{(1 - J_3 k_i)^2}\right]$$

We will next investigate the sign-definiteness of the function $\bar{V}_1(y(z)) = V_2(z)$ in special cases of the asymptotic stability boundary of motion (1.2), in which one or several of the inequalities in (1.3) become equalities.

3. Parts of the asymptotic stability boundaries

We will first examine the case of the stability boundary

$$H = \omega_0[4(A - B) + J_1] \tag{3.1}$$

In this case $m_1 = 0 = M_1$, and the values $m_2 > 0$ and $m_3 > 0$ satisfy the equation

$$f_1(m) = J_1 m^2 - [J_1(A + J_1)\omega_0^2 + AJ_1 + 1]m + A[(J_1^2 + 1)\omega_0^2 + 1] = 0 \tag{3.2}$$

When the quantity (3.1) is substituted into $V_2(z)$, we obtain the function $V_3(z)$. Since for the remaining eigenvalues we have $k_i > 0$ (i = 1, 2, 3), then, according to Theorem 2, the parametric substitution here is the following

$$z_1 = t, z_i = \sum_{j=2}^{\infty} b_{ij} t^j, \quad i = 2, 3, \ldots, 7 \tag{3.3}$$

As a result of substitution (3.3), for $V_3(z)$ we obtain a series in the parameter t that begins with fourth-order terms, so that

$$A_4(b_{i(p)};1;2;t) = \frac{1}{2}(K_3b_{22}^2 + K_1b_{32}^2 + K_2b_{42}^2 + 3\omega_0^2(C-A)b_{52}^2 + M_2b_{62}^2 + M_3b_{72}^2) + D_1$$

where

$$D_1 = \frac{1}{24}\omega_0^2[12A - 11B + 3(J_1 - J_2)]$$

According to the second assertion of Theorem 1, the condition $D_1 > 0$ is necessary and sufficient for the sign-definiteness of $V_3(z)$.

We will examine the case $D_1 = 0$ separately. The only solution of Eq. (2.3) is then

$$b_{22} = b_{32} = \ldots = b_{72} = 0 \tag{3.4}$$

The parametric substitution (3.3) will next be continued by taking into account equalities (3.4). As a result, using the expression $J_2 = 4A - 11B/3 + J_1$, for $V_3(z)$ we obtain a series that begins with sixth-order terms. Now we have

$$A_6(b_{i3};1;3;t) = \frac{1}{2}(K_3b_{23}^2 + K_1b_{33}^2 + K_2b_{43}^2 + 3\omega_0^2(C-A)b_{53}^2)$$
$$+\frac{1}{2}\left[M_2\left(b_{63} + \frac{m_2\omega_0^2}{6M_2}G_{12}\right)^2 + M_3\left(b_{73} + \frac{m_3\omega_0^2}{6M_3}G_{13}\right)^2\right] + D_2$$

where

$$D_2 = \frac{\omega_0^2}{72}[(A + J_1) - G_{12}^2 - G_{13}^2] + \frac{B\omega_0^2}{180}, \quad G_{1i} = \frac{A}{A - m_i} + \frac{J_1^2}{1 - J_1m_i}, \quad i = 2,3$$

The latter expression is simplified when the roots of Eq. (3.2) are substituted into it: $D_2 = B\omega_0^2/180 > 0$. As a result, asymptotic stability condition (1.3) can be supplemented by the following

$$\begin{aligned} &(B - C - J_3)\omega_0 + H > 0, \quad C > A, \quad B \neq A \\ &[4(B - A) - J_1]\omega_0 + H = 0, \quad 12A - 11B + 3(J_1 - J_2) \geq 0 \end{aligned} \tag{3.5}$$

The problem of asymptotic stability is resolved in this case by analysing the terms up to the sixth order in the expansion of the function $V_3(z)$.

We will now examine the part of the asymptotic stability boundary

$$H = \omega_0(C - B + J_3) \tag{3.6}$$

Here $k_1 = 0$, and, therefore, $K_1 = 0$. All the eigenvalues $m_i > 0$ ($i = 1, 2, 3$); therefore, in accordance with Theorem 2, the parametric substitution will have the form

$$z_3 = t, \quad z_i = \sum_{j=2}^{\infty} b_{ij}t^j, \quad i = 1,2,4,\ldots,7 \tag{3.7}$$

As a result of substituting expressions (3.6) and (3.7) into the function $V_2(z)$, we obtain the series $V_4(z)$ in the parameter t, which begins with fourth-order terms, so that

$$A_4(b_{i2};1;2;t) = \frac{1}{2}(M_1b_{12}^2 + K_2b_{22}^2 + K_3b_{32}^2 + 3\omega_0^2(C-A)b_{52}^2 + M_2b_{62}^2 + M_3b_{72}^2) + D_3$$

where

$$D_3 = \frac{1}{24}\omega_0^2[3C - 2B + 3(J_3 - J_2)]$$

According to the second assertion of Theorem 1, satisfaction of the condition $D_3 > 0$ is necessary and sufficient for sign-definiteness of the function $V_4(z)$.

An analysis of the case when $D_3 = 0$ also leads to the value

$$A_6(b_{i3};1;3;t) = \frac{1}{2}(M_1b_{13}^2 + M_2b_{63}^2 + M_3b_{73}^2 + 3\omega_0^2(C-A)b_{53}^2)$$
$$+\frac{1}{2}\left[K_3\left(b_{23} + \frac{k_3\omega_0^2}{6K_3}G_{23}\right)^2 + K_2\left(b_{43} + \frac{k_2\omega_0^2}{6K_2}G_{22}\right)^2\right] + \frac{B\omega_0^2}{180}$$

where

$$G_{2i}=\frac{C}{C-k_i}+\frac{J_3^2}{1-J_3k_i}, i=2,3$$

Therefore, according to Theorem 1, in this case, too, the function V_4 is positive-definite. Thus, the conditions for asymptotic stability of steady solution (1.2) on part (3.6) of the boundary are expressed as follows:

$$\begin{aligned}&[4(B-A)-J_1]\omega_0+H>0,\quad B\neq A,\quad C>A,\\&(B-C-J_3)\omega_0+H=0,\quad 3C-2B+3(J_3-J_2)\geq 0\end{aligned}\tag{3.8}$$

Here also the question of asymptotic stability is resolved by terms no higher than the sixth order in the expansion of the function V_4.

We will analyse how the stability boundaries are reached when

$$\omega_0[4(A-B)+J_1]=H=\omega_0(C-B+J_3)$$

In this case $m_1=0=k_1$, and accordingly $M_1=0=K_1$. According to Theorem 2, the parametric substitution will be

$$z_1=t_1,\quad z_3=t_3,\quad z_i=\sum_{l_1+l_3=2}^{\infty}b_{i(l_1,l_3)}t_1^{l_1}t_3^{l_3};\quad l_1,l_3\geq 0,\quad i=2,4,\ldots,7,\quad b_{i(l_1,l_3)}\in R\tag{3.9}$$

When expressions (3.9) are substituted into the expression for $V_3(z)$, we obtain a series in t_1 and t_3 that begins with fourth-order terms, so that

$$A_4(b_{i(l_1,l_3)};1;2;t_1,t_3)=\frac{1}{2}[t_1^4P_1(2,0)+t_3^4P_1(0,2)]+t_1^3t_3P_2(2,0)+t_1t_3^3P_2(0,2)+t_1^2t_3^2P_3$$

where

$$\begin{aligned}&P_1(i,j)=K_2b_{4(i,j)}^2+K_3b_{2(i,j)}^2+M_2b_{6(i,j)}^2+M_3b_{7(i,j)}^2+3\omega_0^2(C-A)b_{5(i,j)}^2\\&+\omega_0^2\left[A-\frac{11}{12}B+\frac{1}{4}(J_1-J_2)\right]\\&P_2(i,j)=K_2b_{4(i,j)}b_{4(1,1)}+K_3b_{2(i,j)}b_{2(1,1)}+M_2b_{6(i,j)}b_{6(1,1)}+M_3b_{7(i,j)}b_{7(1,1)}\\&+3\omega_0^2[(C-A)b_{5(i,j)}b_{5(1,1)}+(B-C)b_{5(i,j)}]\\&P_3=K_2\left[\frac{1}{2}b_{4(1,1)}^2+b_{4(2,0)}b_{4(0,2)}\right]+K_3\left[\frac{1}{2}b_{2(1,1)}^2+b_{2(2,0)}b_{2(0,2)}\right]\\&+M_2\left[\frac{1}{2}b_{6(1,1)}^2+b_{6(2,0)}b_{6(0,2)}\right]+M_3\left[\frac{1}{2}b_{7(1,1)}^2+b_{7(2,0)}b_{7(0,2)}\right]\\&+3\omega_0^2\left[(C-A)\left\langle\frac{1}{2}b_{5(1,1)}^2+b_{5(2,0)}b_{5(0,2)}\right\rangle+(B-C)b_{5(1,1)}\right]+\omega_0^2\left[A-\frac{9}{4}B+\frac{3}{2}C+\frac{1}{4}(J_1-J_2)\right]\end{aligned}$$

Determination of the region of positive values of $A_4(b_{i(l_1,l_3)};1;2;t)$ involves directly finding the smallest value of this quantity. For the function A_4, which depends on 15 variables $b_{i(l_1,l_3)}$ $(i=2,4,5,6,7;l_1,l_3=0,1,2)$, this smallest value is

$$D_4=\frac{1}{24}\omega_0^2\left\{[3C-2B+3(J_3-J_2)](t_1^4+t_3^4)+6\left[7C-6B+(J_3-J_2)-6\frac{(C-B)^2}{C-A}\right]t_1^2t_3^2\right\}$$

It is calculated by making the substitution $J_1=J_3+C+3B-4A$ and is specified when

$$\begin{aligned}&t_1^2b_{i(2,0)}+t_1t_3b_{i(1,1)}+t_3^2b_{i(0,2)}=0,\quad i=2,4,6,7\\&t_1^2b_{5(2,0)}+t_1t_3b_{5(1,1)}+t_3^2b_{5(0,2)}=\frac{C-B}{C-A}t_1t_3\end{aligned}\tag{3.10}$$

The biquadratic form for the function D_4 of two variables has the form

$$F(t)=M_1t_1^4+2M_2t_1^2t_3^2+M_1t_3^4$$

where

$$M_1=3C-2B+3(J_3-J_2),\quad M_2=3\left[7C-6B+J_3-J_2-6\frac{(C-B)^2}{C-A}\right]$$

and will be positive-definite when one of the following set of conditions is satisfied

$$1)\ M_1\geq 0,\quad M_2>0;\quad 2)\ M_1>0,\quad -M_1<M_2<0$$

They can be expressed in terms of the moments of inertia of the system of bodies in the form

$$J_3 - J_2 \geq \frac{2}{3}B - C, \quad J_3 - J_2 > 6B - 7C + 6\frac{(C-B)^2}{C-A} \tag{3.11}$$

$$J_3 - J_2 > \frac{2}{3}B - C, \quad \frac{10}{3}B - 4C + \frac{(C-B)^2}{C-A} < J_3 - J_2 < 6B - 7C + 6\frac{(C-B)^2}{C-A} \tag{3.12}$$

It is especially urgent to investigate the case of the form of constant sign $F(t)$, in which the first inequality sign in the second condition in (3.12) is replaced by an equality sign. Then parametric substitution (3.9) will be continued:

$$z_1 = t_1, \quad z_3 = t_3, \quad z_5 = \frac{C-B}{C-A}t_1t_3 + \sum_{l_1+l_3=3}^{\infty} b_{5(l_1,l_3)}t_1^{l_1}t_3^{l_3}$$

$$z_i = \sum_{l_1+l_3=3}^{\infty} b_{i(l_1,l_3)}t_1^{l_1}t_3^{l_3}; \quad l_1, l_3 \geq 0, \quad i = 2,4,6,7, \quad b_{i(l_1,l_3)} \in R \tag{3.13}$$

As a result of substituting expressions (3.13) into the function $V_3(z)$, we obtain a series for which the expansion begins with fifth-order terms:

$$A_5(b_{i(l_1,l_3)};1;3;t) = 3\omega_0^2t_1t_3[t_1^3b_{5(3,0)} + t_1^2t_3b_{5(2,1)} + t_1t_3^2b_{5(1,2)} + t_3^3b_{5(0,3)}]$$

In the general case of arbitrary real $b_{5(l_1,l_3)}$, the latter expression does not vanish identically. According to Theorem 1, the form $V_3(z)$ is sign-variable. Then, according to the Barbashin and Krasovskii theorems,[2,3] motion (1.2) is unstable.

Therefore, under condition (3.10) and either conditions (3.11) or conditions (3.12), steady motion (1.2) is asymptotically stable. Here the question of asymptotic stability is resolved by analysing the terms up to the sixth order in the expansion of the function $V_3(z)$.

We will investigate the part of the asymptotic stability boundary for $C = A$. In this case the matrices S_1 and S_2 of the quadratic forms are non-degenerate, and, therefore, by Theorem 2, the parametric substitution for $C = A$ will be the following

$$z_5 = t, \quad z_i = \sum_{j=2}^{\infty} b_{ij}t^j, \quad i = 1,2,3,4,6,7 \tag{3.14}$$

As a result of substituting expressions (3.14) into the function $V_2(z)$, we obtain a series, whose expansion in the parameter t begins with the smallest (fifth) order:

$$A_5(b_{i2};1;2;t) = 3(B-A)\omega_0^2(b_{12} + b_{62} + b_{72})(b_{22} + b_{32} + b_{42})$$

When $A \neq B$ (the case of a spherical satellite is not considered here), the conclusion that the form $V_2(z)$ is sign-definite hence follows according to the first assertion of Theorem 1, regardless of the signs of the expressions in parentheses. Thus, in this case, according to the Barbashin and Krasovskii theorems,[2,3] motion (1.2) is unstable. Violation of the condition $C > A$ in any part of the asymptotic stability boundary results at once in unstable motions of the form (1.2).

When $A \neq B$, in the case in which only the left-hand side of the first strict inequality in (1.3) vanishes and conditions (3.5) are satisfied, motion (1.2) is asymptotically stable.

When $A \neq B$, in the case in which only the left-hand side of the second strict inequality in (1.3) vanishes and conditions (3.8) are satisfied, motion (1.2) is asymptotically stable.

When $A \neq B$, in the case in which the left-hand sides of the first and second strict inequalities in (1.3) vanish and either conditions (3.11) or conditions (3.12) are satisfied, steady motion (1.2) is also asymptotically stable.

Obviously, when $A = B$, according to the Barbashin and Krasovskii theorems,[2,3] motion (1.2) will not be asymptotically stable in any part of the boundary. At best, according to the corresponding Lyapunov theorem,[2,3] only stability of motion (1.2) can be obtained.

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